

# Flow properties for the continuum viewpoint of a non-equilibrium gas-particle mixture

By RONALD PANTON

Department of Mechanical and Aerospace Engineering  
Oklahoma State University, Stillwater, Oklahoma

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Flow properties for the non-equilibrium two-phase flow of a gas-particle mixture are formulated from the theoretical standpoint. A quasi-one-dimensional flow containing an arbitrary volume of particles is considered, and mass transfer between the phases is allowed. It is shown that meaningful definitions of the flow properties of each phase can be constructed as area-averages of (time-averaged local flow-field properties). Special definitions of averages overcome the difficulties introduced by the fact that one phase does not occupy the entire region at all times. Conservation equations for the newly defined properties are given and criteria for their validity determined. The results give fresh interpretation to several aspects of two-phase flow: the particle-phase pressure is associated with the internal particle pressure, whereas Reynolds-stress terms are introduced by fluctuations in particle velocity. Reynolds stresses for both phases are important in laminar as well as turbulent flow and provide a significant particle-phase viscous effect. The interphase momentum transfer because of condensation or vaporization is shown to be characterized by the particle-phase velocity irrespective of the direction of the mass transfer.

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## 1. Introduction

Fluid-mechanic problems involving gas-particle mixtures arise in many processes of practical importance. One of the earliest problems is that of the heat and mass transfer in the mist-flow region of a boiler tube. The liquid rocket is another example. Usually the oxidizer vaporizes much more rapidly than the fuel spray and combustion occurs heterogeneously around each droplet. The length of the combustion chamber and the stability of the flow to acoustic or shock waves are practical two-phase flow problems.

Solid rockets also have important particle-gas flow problems. Metal particles enriching the mixture are carried through the combustion chamber and out of the nozzle. These particles also burn heterogeneously and may actually be liquid droplets in the latter stages. The disequilibrium between the velocity of the gases and the burning particles is an important loss of thrust. In aircraft design problems of rain or hail impingement on surfaces arise. Removal of rain from a windshield and avoidance of particle ingestion in jet intakes during taxi are other practical problems. The writer is not familiar with any problems on the astronomical scale; however, phenomena such as the solar wind demonstrate that

space contains ample matter to be considered as a continuum for certain problems. There are, of course, many examples of dispersed particles: meteoric showers, asteroid belts, star clusters, galaxies, comets, the rings of Saturn, etc.

Fluidized beds are an important innovation in chemical engineering. In the simplest arrangement a screen supports particles, perhaps a catalyst, in a vertical tube. The gas flows from bottom to top, and, when a sufficient velocity is reached, the bed becomes fluidized, and the particles are supported individually by the drag of the gas stream. This type of particle-gas flow is non-dilute in that the particles occupy a significant portion of the flow cross-section. Pertinent problems in operating the beds deal with stability, expansion, particle segregation, 'boiling' of gas bubbles, etc. Many variations of fluidized beds are possible: a screen may restrain the particles at the top in addition to the bottom; particles may be extracted at the bottom and continuously fed in at the top so that the bed moves; beds which move upward are also used. Sometimes the purpose of the system is to transfer heat, utilizing the high heat capacity of the particles. Another variation is the spouting bed, where the gas enters a relatively small hole and the sides of the bed are conical. This leads to a central, upward-moving, jet with recirculation of the particles around the sidewalls. Such an arrangement could be used for a batch-drying process.

Transportation of particle materials, such as wheat, by entrainment in a gas flow is another example of purposeful two-phase flow. Sometimes the results are undesirable as in the case of dirt particles in the flow from a gas well. An interesting scheme for transport of fine particles such as cement is to construct a pipe of two rectangular chambers separated by a porous wall. Gas flows into the lower chamber through the porous wall and fluidizes the particle material in the upper chamber. The top of the upper chamber is a porous cloth to allow the gas to exist but retain the particles. Giving the pipe a tilt allows gravity forces to transport the cement. This system is literally a sliding fluidized bed.

A final example is the flow of blood. Many studies have considered blood as a single fluid with non-Newtonian viscosity. The true nature is a mixture of blood cells and a plasma, a two-phase mixture of particles and liquid. This aspect is undoubtedly important in many blood-flow problems, and may even be significant in unsteady arterial flow.

One finds in the literature an amazing number of derivations of equations for the flow of a gas-particle mixture. The fact that each researcher reformulates the equations for his particular problem is perhaps symptomatic of a lack of clarity about flow properties. The equations have been developed by several authors for various special problems and under various assumptions. A few derivations, primarily from the aeronautical engineering literature, are listed here: Carrier (1958), Van Deemter & Van der Laan (1961), Kliegel (1963), Lype (1965), Marble (1964), Murray (1965), Rannie (1962), Rudinger (1965), Soo (1961), Tien (1961), Williams (1959) and Zuber (1964).

This paper will be concerned with the flow properties of a non-equilibrium two-phase system. The condensed phase is assumed to be dispersed droplets which are imbedded in a gas. Allowance must be made for disequilibrium between the velocity and temperature of the particles and those of the gas phase. Hence it is

necessary to account for mass, momentum, and energy exchanges between the two phases.

The conservation equations governing the two-phase flow are usually derived by physical arguments. These derivations are essentially applications of conservation concepts to a control volume. They presuppose the definition of velocities, pressures, temperatures, etc., for each phase. Terms which represent various effects are formulated by physical arguments and analogy with similar terms for single-phase flow. This approach actually amounts to a rederivation of the laws of fluid motion for the two-phase system. In such a derivation, the worker is obliged to formulate each term properly and include all significant factors.

An alternative approach, which is more rigorous and deductive in nature, is presented. Starting with the local conservation equations, one may proceed mathematically by integrating over a control volume, defining new average variables and limiting the control volume to zero size. Subsequent time averaging will produce new differential equations describing the two-phase flow. The advantage of such an approach is two-fold. First, the variables appearing in the final equations have explicit definitions in terms of the local flow-field variables. Secondly, the assumptions are stated explicitly during the derivation in order to simplify the mathematical expressions and thus the reason, necessity and effect of each assumption is apparent. Together they lead to a clearer understanding of the meaning of each property in the two-phase equations and of the applicability of the equations themselves.

This paper will deal with quasi-one-dimensional flow. Some of the problems noted above require a full three-dimensional treatment and hence the results are not directly applicable in these cases. The purpose is to present a formalism where the flow is viewed as two co-existent and interacting continua. Emphasis is placed upon the proper definition of the flow properties, the conservation equations relating these properties, and the criteria under which the equations are valid. The method gives a fresh interpretation of the physical role and origin of each property and in some cases a completely different interpretation than that of previous work.

## **2. Physical background**

In order to simplify the work and leave the central ideas unencumbered with details, the flow is considered as one-dimensional without the presence of any body or electrical forces. The particles are either liquid droplets or solid particles, but they do not break up or shatter and are incompressible. The discussion will be carried out for liquid droplets realizing that the equations are the same or simpler for solid particles. Also, the case of solid particles carried by a liquid matrix instead of a gas requires only simple changes; however, compressible particles, such as gas bubbles in a liquid, would require an extensive reformulation. Mass may be transferred to or from the particles by condensation or evaporation; however, all particles at a particular position and time are the same size.

An important restriction is that the diameter of the particle is a macroscopic length. This is necessary so that it is meaningful to form integrals over the particle surface of local continuum properties.

One usual assumption which is not made here is that the mixture is dilute in the sense that the volume of the particles is a small fraction of the total volume. A fluidized bed is an example of a non-dilute flow. Another facet of the present investigation is that we specifically include turbulence.

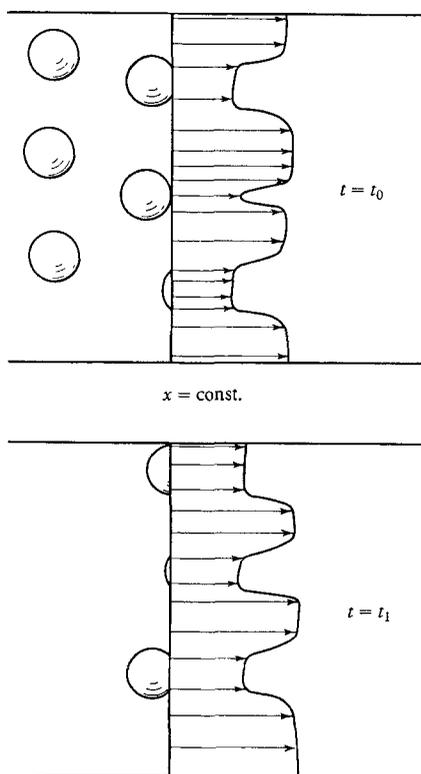


FIGURE 1. Flow field.

The interface between the gas and the particle material is idealized as a surface of zero thickness which contains no mass, momentum, or energy. On one side of the surface the thermodynamic properties are those of the gas phase, and on the other, those of the particle material. The temperature and tangential velocity are continuous across the interface surface. The surface does have one property, a tension force which maintains the mechanical equilibrium of the droplet. Another assumption is that particles are large enough so that the thermodynamic properties are independent of the radius. Under usual circumstances this means that the particles must be larger than about  $1\mu$  in diameter.

The flow field is represented schematically in figure 1 where the velocity across a typical cross-section is shown. The gas velocity is arbitrarily assumed higher than that of the liquid particles. As the surface of a particle is approached the velocity drops so that gas sticks to the surface of the droplet. At positions inside the droplet the velocity is the liquid velocity. The local gas velocity is denoted by  $v_g$  and the local liquid (or solid) velocity by  $v_p$ . The temperature and density fields are similar to the velocity field. One difference is that the density is dis-

continuous at the surface: constant at  $\rho_l$  on the liquid side, and equal to  $\rho_g$  on the gas side. As time proceeds, the particles move past the cross-section and positions previously occupied by liquid are now occupied by gas. The velocity and all other properties have changed appropriately. From these considerations we see that the two-phase flow is essentially unsteady and three-dimensional in its detailed structure.

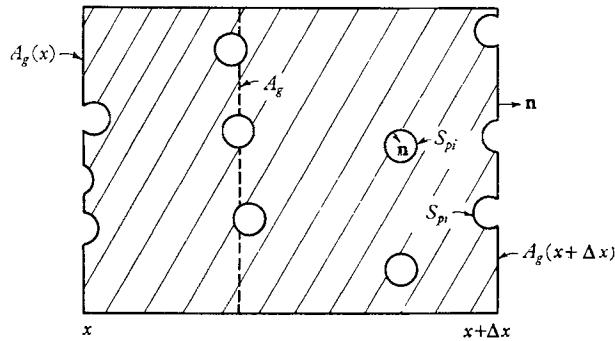


FIGURE 2. Gas-phase control volumes.

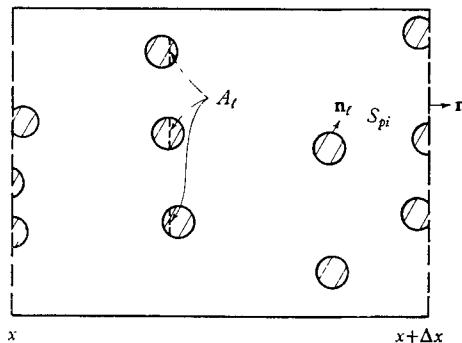


FIGURE 3. Particle-phase control volume.

The usual differential equations expressing the conservation laws are valid throughout the entire flow field of course, but as already noted, it is impractical to seek a detailed solution. Instead, the flow will be described in terms of averages. It is also desired to obtain differential equations which relate the averages. These equations then express conservation principles for the average properties. Actually the process is turned around; the local conservation laws will be integrated over a control volume (and eventually time-averaged), and then averages defined so that they are a simple term in the integrated conservation law. For instance, a typical question to be answered would be how should the particle pressure be defined. To answer this we will look at the integrated form of the momentum conservation law and define the particle pressure so that it is a force which changes the momentum of the particle material.

Figures 2 and 3 illustrate several choices of control volumes which are of interest in two-phase flow. The one-dimensional flow is in the  $x$ -direction and a circular cross-section of area  $A$  is considered. The gas-phase control volume is

defined as the region between end planes at  $x$  and  $x + \Delta x$ , which is occupied by gas. This control volume resembles a piece of Swiss cheese where the holes represent regions occupied by the particles. The planes  $x$  and  $x + \Delta x$  are stationary but the holes move with time. Gas enters and leaves by flow through the end planes and also by condensation or vaporization through the holes. In the derivations, it will be necessary to integrate over the region of the control volume. This will be done by first integrating over the 'gas cross-section'  $A_g$  and then over the  $x$ -direction. The region  $A_g$  is the portion of the cross-section occupied by the gas; that is, the total cross-section  $A$  excluding the holes for the particles. A consistent nomenclature is used where all regions which are cross-sections perpendicular to the flow are denoted by  $A$  and all particle surfaces are denoted by  $S$ .

The particle-phase control volume consists of the sum of disjoint regions between  $x$  and  $x + \Delta x$  which are occupied by particle material. This region consists of many (a finite number) spheres which lie completely inside the planes at  $x$  and  $x + \Delta x$  plus the truncated portions of spheres which are intersected by these end planes. The spherical portions are moving at the appropriate particle velocity. The flat circular portions at the end planes do not move in the  $x$ -direction but may be increasing or decreasing in size depending on whether the particle is approaching or leaving the end plane. Liquid or particle material enters or leaves the particle control volume by flow across the circular portions at the end planes. Vaporization or condensation may also occur across the particle surfaces. Again it is convenient to define a particle or liquid-phase cross-section  $A_\ell$  as the portion of the total cross-section occupied by particle material. The sum of  $A_g$  and  $A_\ell$  is the total cross-section  $A$ .

The mixture control volume is the entire region between  $x$  and  $x + \Delta x$  with cross-section  $A$ . It is simply the sum of the gas and particle phase control volumes. The region is fixed in space and both gas and particle material enter and leave through the end planes.

### 3. Properties for the gas-phase continuity equation

The properties appearing in the continuity equation for the gas phase will be discussed in this section. The derivation is performed by applying the integral form of the continuity equation to the gas-phase control volume. A differential equation is then obtained by introducing definitions of area-averaged properties and limiting the length of the control volume to zero. Next, the equation is time-averaged because of the unsteady nature of the flow. The final step is to show that the time-average of (an area-averaged property) is equal to the area-average of (a local time-averaged property). When this is done the variables in the gas-phase continuity equation are well defined in terms of averages of the local flow field variables.

The integral form of the continuity equation for a control volume  $V(t)$  with surface  $S(t)$  moving with an arbitrary velocity  $\mathbf{v}_s$  is given by Thorpe (1962),

$$\frac{d}{dt} \int_{V(t)} \rho dV = - \int_{S(t)} \rho (\mathbf{v} - \mathbf{v}_s) \cdot \mathbf{n} dS. \quad (3.1)$$

In this equation  $\rho$  and  $\mathbf{v}$  are the local density and velocity while  $\mathbf{n}$  is the outward normal vector from  $S$ . Applying the equation to the gas-phase control volume gives

$$\frac{d}{dt} \int_x^{x+\Delta x} \int_{A_g(x,t)} \rho_g dA dx = - \int_{A_g(x,t)} \rho_g \mathbf{v}_g \cdot \mathbf{n} dA \Big|_x^{x+\Delta x} - \sum_i \int_{S_i} \rho_g (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} dS.$$

The left-hand side is the rate of accumulation of matter within the control volume. The first term on the right-hand side is the net flow into the control volume through the ends at  $x$  and  $x + \Delta x$ . The corresponding integrals over the sides are taken to be zero because of the assumed one-dimensional nature of the flow. The second term is the summation over the individual particles of the gas flux into the gas phase by vaporization or condensation.

The equation is simplified in the following manner. On the left-hand side the time derivative is taken inside the integration over  $x$  and the integral is then estimated by the mean-value theorem for integrals. Thus

$$\frac{d}{dt} \int_x^{x+\Delta x} \int_{A_g(x,t)} \rho_g dA dx = \Delta x \frac{\partial}{\partial t} \int_{A_g(x_0,t)} \rho_g dA,$$

where  $x_0$  must lie between  $x$  and  $x + \Delta x$ . Next, an area average is defined and introduced into the first two terms

$$\|f\| (x, t) \equiv \frac{1}{A_g} \int_{A_g} f(x, y, z, t) dA. \tag{3.2}$$

(Here and subsequently, vertical strokes are used to denote area averages, while time averages are distinguished by the conventional symbols.) The continuity equation is divided by  $\Delta x$  and the limit  $\Delta x \rightarrow 0$  is taken. During this process  $x_0$  takes on the value  $x$ . Then

$$\frac{\partial}{\partial t} (A_g \|\rho_g\|) = - \frac{\partial}{\partial x} (A_g \|\rho_g v_{gx}\|) + \psi. \tag{3.3}$$

In this equation  $\psi$  represents the rate at which mass is added to the gas phase from the particles per unit length of flow. Therefore

$$\psi \equiv - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \sum_i \int_{S_{pi}} \rho_g (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} dS. \tag{3.4}$$

The flow field is inherently unsteady, and it is therefore necessary to time-average (3.3) in a method similar to turbulent-flow theory. It will be necessary to use only some elementary concepts as discussed in chapter 1 of Hinze's (1959) book. The time-average of a variable which is a continuous function of  $x$  and  $t$  only is defined as follows:

$$\langle\langle f \rangle\rangle (x, t) \equiv \frac{1}{t_0} \int_t^{t+t_0} f(x, t + \xi) d\xi. \tag{3.5}$$

The average is assumed independent of further averaging

$$\langle\langle\langle\langle f \rangle\rangle\rangle\rangle = \langle\langle f \rangle\rangle.$$

Other properties of the average are

$$\begin{aligned}\langle\langle f+g \rangle\rangle &= \langle\langle f \rangle\rangle + \langle\langle g \rangle\rangle, & \langle\langle\langle\langle f \rangle\rangle \cdot g \rangle\rangle &= \langle\langle f \rangle\rangle \cdot \langle\langle g \rangle\rangle, \\ \langle\langle\left(\frac{\partial f}{\partial x}\right)\rangle\rangle &= \frac{\partial}{\partial x} \langle\langle f \rangle\rangle, & \langle\langle\left(\frac{\partial f}{\partial t}\right)\rangle\rangle &= \frac{\partial}{\partial t} \langle\langle f \rangle\rangle.\end{aligned}$$

The last equation is true if the unsteady phenomena has a characteristic time considerably longer than the time required to form suitable averages  $t_0$ . The fluctuation  $f''$  of a function of  $x$  and  $t$  is defined by the following equation:

$$f = \langle\langle f \rangle\rangle + f''. \quad (3.6)$$

The time-average of  $f''$  is zero and the following useful relation may also be proved:

$$\langle\langle fg \rangle\rangle = \langle\langle f \rangle\rangle \langle\langle g \rangle\rangle + \langle\langle f'' g'' \rangle\rangle. \quad (3.7)$$

Returning to the continuity equation (3.3), we integrate from  $t$  to  $t+t_0$  and divide by  $t_0$  to obtain the result

$$\frac{\partial}{\partial t} \langle\langle A_\sigma \| \rho_\sigma \| \rangle\rangle = -\frac{\partial}{\partial x} \langle\langle A_\sigma \| \rho_\sigma v_{\sigma x} \| \rangle\rangle + \langle\langle \psi' \rangle\rangle. \quad (3.8)$$

It is possible to apply (3.7) to the product terms in the equation above and obtain an equation where the fluctuations appear explicitly. As an example, the term on the left-hand side would be

$$\frac{\partial}{\partial t} \langle\langle A_\sigma \| \rho_\sigma \| \rangle\rangle = \frac{\partial}{\partial t} (\langle\langle A_\sigma \rangle\rangle \langle\langle \| \rho_\sigma \| \rangle\rangle + \langle\langle A_\sigma'' \| \rho_\sigma'' \| \rangle\rangle).$$

In this equation the variables are first averaged across the area perpendicular to the flow and then time-averaged. The assumption that the particles are dispersed has not yet been used, and (3.8) is valid for any flow, dispersed or not. From an experimental standpoint, the density profile  $\rho_\sigma$  must be measured across the section at one instant in time; such a detailed knowledge would be hard to obtain. Then the instantaneous area-averaged density could be computed. A continuous series of measurements would be required in order to compute the time average. This will not be pursued since it is desirable to define the properties of a two-phase flow in the opposite way; that is, the area average of a time-averaged local property. This property would be constructed by obtaining first the local time averaged values, a concept already familiar from turbulent flow and more easily measured than instantaneous properties. Performing an area-averaging then gives the final property. Another reason for this approach is that the local Reynolds stresses, a well-established concept, will come out of the development for the momentum equation. If the order of averaging were not interchanged, this identification could not be made. In making the interchange of the order of averaging we will find that the dispersed flow assumption is necessary.

An example of the appropriate definition of gas-phase velocity is

$$V_{gx} \equiv \| \langle\langle v_{gx} \rangle\rangle \|.$$

This relation has a defect however. The local gas velocity  $v_{gx}$  is only defined for times when gas occupies the position  $x, y, z$  but the time average  $\langle\langle \rangle\rangle$  was restricted to apply to functions defined for all  $x$  and  $t$ . This defect is corrected by introducing another time-average process for functions which are not defined for all  $x, y, z$  and  $t$ .

The local instantaneous gas velocity is shown in figure 4 as a function of time. When a particle occupies the particular position,  $v_{gx}$  is undefined. A local time-average for such a function can be defined as

$$\langle f_g \rangle (x, y, z, t) \equiv \frac{1}{t_{0g}} \int_t^{t+t_0} f_g(x, y, z, t + \xi) d\xi. \tag{3.9}$$

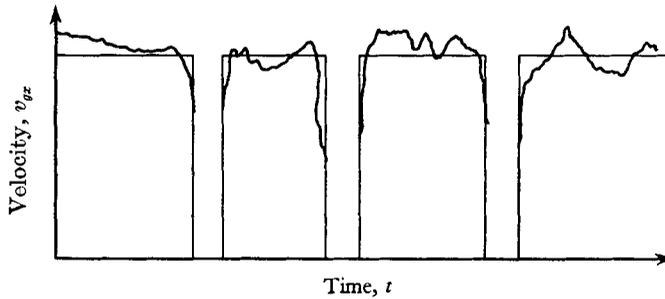


FIGURE 4. Velocity history.

The integrand is now taken to be zero when a particle passes  $x, y, z$  and the time interval  $t_{0g}$  is that portion of  $t$  to  $t + t_0$  when gas occupied the point  $x, y, z$ . Also,  $\langle f_g \rangle$  is considered to be defined only in the regions where  $f_g$  is defined. The new definition actually includes the old  $\langle\langle \rangle\rangle$  average as a special case. All the relations given previously for the  $\langle\langle \rangle\rangle$  average also hold true for the new time average so long as  $f_g$  and  $g_g$  are defined in the same regions of time and space. The local fluctuation is denoted by a single prime. Then

$$f(x, y, z, t) = \langle f \rangle + f'. \tag{3.10}$$

For the product of two functions an equation analogous to (3.7) is

$$\langle f_g \rangle = \langle f \rangle \langle g \rangle + \langle f'g' \rangle.$$

In the same spirit it is possible to denote the deviation of a property from the area-averaged value defined earlier in (3.2):

$$f(x, y, z, t) = \|f\| (x, t) + f^*(x, y, z, t), \tag{3.11}$$

the deviation being denoted by  $f^*$ .

In order to convert (3.8) into a more meaningful and useful form, we want to interchange the order of the time and area-averaging processes. Thus, we seek to prove the relation

$$\langle\langle \|f\| \rangle\rangle = \|\langle f \rangle\|. \tag{3.12}$$

In order to derive this, we first consider a particular cross-section of the flow field. The part of the cross-section occupied by gas is denoted by  $A_g(x, t)$ . As time proceeds, different portions of the cross-section are occupied by particles, as the

original particles have moved downstream and new particles have moved into the cross-section from upstream. Figure 5 shows  $A_g(x, t)$  for a particular choice of  $x$  and for a time interval  $t$  to  $t + t_0$ . The black regions in figure 5 represent the passage of a particle past the cross-section. The time  $t_{0g}(x, y, z, t)$  is the length from the front face of the cylinder to the back face excluding the distance through the 'particles'. The region in the cylinder excluding the black portions will be called  $R$ ; it is the domain in  $y, z, t$  when gas properties are defined. An important auxiliary equation can be obtained by computing the volume of  $R$  in two different ways:

$$\text{vol } R = t_0 \langle\langle A_g \rangle\rangle = A \|t_{0g}\|.$$

An additional relation is obtained by computing the integral of any gas-phase property  $f$  by two different methods

$$t_0 \langle\langle A_g \|f\| \rangle\rangle = A \|t_{0g} \langle f \rangle\|. \tag{3.13}$$

By using the two equations above together with (3.7) and (3.10), one may show that

$$\langle\langle \|f\| \rangle\rangle + \frac{\langle\langle A_g'' \|f\|'' \rangle\rangle}{\langle\langle A_g \rangle\rangle} = \| \langle f \rangle \| + \frac{\langle\langle t_{0g}^* \langle f \rangle^* \rangle\rangle}{\|t_{0g}\|}.$$

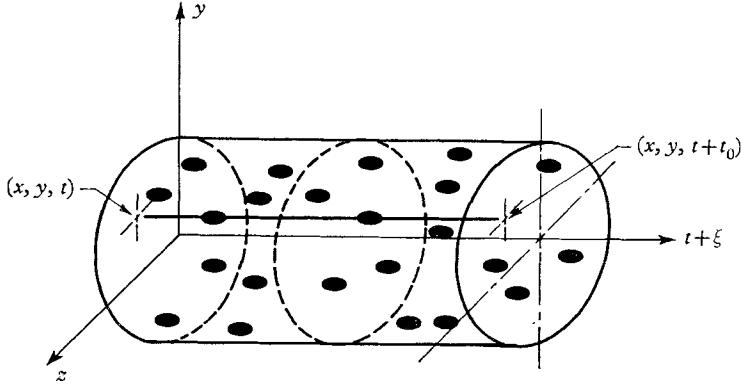


FIGURE 5. History at a cross-section.

If the second term on each side of this equation is zero, then we have the desired relation given by (3.12).

The second term on the left-hand side could be neglected if the fluctuation in the area is much smaller than the average area

$$A_g'' / \langle\langle A_g \rangle\rangle \simeq 0. \tag{3.14}$$

Since the sum of the liquid and gas cross-sections is the total flow area, a constant, the fluctuations  $A_g''$  are actually compensations for fluctuations  $A_l''$  caused by the particles. If the number of particles within the cross-section  $A_g$  is much larger than the fluctuation in the number of particles, then the condition above could be expected to hold. The second term on the right-hand side can be neglected if

$$t_{0g}^* / \|t_{0g}\| \simeq 0. \tag{3.15}$$

Again this is related to the presence or absence of a particle at a point during the time-averaging process. If the particles are numerous and dispersed, then they will pass the point at regular intervals throughout the cross-section, and the deviation  $t_{0g}^*$  will be much smaller than the area-averaged value. In terms of

figure 5, these equations are true if the cylinder contains numerous black dots dispersed uniformly without holes or voids or bunches, i.e. in a continuous fashion. There is an analogy here with the continuum hypothesis in fluid mechanics that molecules are dispersed in space in a uniform manner.

The inversion of area and time averaging, as in (3.12) is possible when (3.14) and (3.15) are true. On the other hand, we may stipulate that (3.14) and (3.15) are criteria to be met in order to take a continuum viewpoint of the two-phase flow. In words, average properties of a two-phase flow may be formulated over an area and during a time interval if there are numerous dispersed particles within this time-space region. It should be noted that the control volume cross-section can be chosen arbitrarily and does not necessarily end at a physical wall.

We can now conclude the derivation of the continuity equation. By the assumption above, fluctuations in the gas cross-section  $A'_g$  are negligible. Then

$$A_g = \langle\langle A_g \rangle\rangle + A'_g \simeq \langle A_g \rangle.$$

Substituting this relation, (3.10) for the local time averages and inverting according to (3.12) yields the continuity equation:

$$\frac{\partial}{\partial t} (\langle A_g \rangle \langle \rho_g \rangle) = - \frac{\partial}{\partial x} \{ \langle A_g \rangle ( \langle \rho_g \rangle \langle v_{gx} \rangle_\rho + \langle \rho'_g v'_{gx} \rangle ) \} + \langle \psi \rangle, \quad (3.16)$$

where

$$\langle v_{gx} \rangle_\rho \equiv \frac{1}{\langle \rho_g \rangle} \langle \rho_g v_{gx} \rangle.$$

The second term on the right is the product of the local density and velocity fluctuations. It is usually assumed in turbulence work that fluctuations in the density are negligible compared to the average density. Making this assumption here reduces the equation above to the usual form for one-dimensional flow except that an additional term  $\langle\langle \psi \rangle\rangle$  accounts for matter added to the gas phase from the particle phase. In order to make the notation compact, the averaged properties of the two-phase flow may be denoted by  $\sim$ . Then (3.16) is written

$$\frac{\partial}{\partial t} (\bar{A}_g \bar{\rho}_g) = - \frac{\partial}{\partial x} (\bar{A}_g \bar{\rho}_g \bar{v}_g) + \bar{\psi}, \quad (3.17)$$

where the gas density  $\bar{\rho}_g$  is well defined by time-averaging the local gas density at each point and then performing an area average. Likewise, a 'mass-averaged' velocity  $\bar{v}_g$  is well defined by time-averaging the local gas velocity, multiplying by the density and averaging across an area.

We conclude this section by summarizing the major assumptions under which (3.17) is valid. First, the flow is quasi one-dimensional with particles of a finite size. The flow is dispersed in that the criteria given by (3.14) and (3.15) hold. The area  $A$  and time  $t_0$  chosen for the averaging processes are arbitrary as long as (3.14) and (3.15) are satisfied. Finally, it is necessary that fluctuations in the gas density are negligible.

#### 4. Properties defined by the momentum equation for the gas phase

The momentum equation is manipulated in a manner similar to that used for the continuity equation in the preceding section. The integral form of the momentum equation is

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV = - \int_{S(t)} \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_s) \cdot \mathbf{n} dS + \int_{S(t)} \boldsymbol{\sigma} \cdot \mathbf{n} dS. \quad (4.1)$$

The stress tensor is denoted by  $\boldsymbol{\sigma}$  in this equation. When the  $x$ -component of this equation is applied to the gas-phase control volume, one obtains

$$\begin{aligned} \frac{d}{dt} \int_x^{x+\Delta x} \int_{A_g} \rho_g v_{gx} dA dx = & - \int_{A_g} \rho_g v_{gx} (\mathbf{v}_g \cdot \mathbf{n}) dA \Big|_x^{x+\Delta x} + \int_{A_g} (\boldsymbol{\sigma}_g \cdot \mathbf{n})_x dA \Big|_x^{x+\Delta x} \\ & - \sum_i \int_{S_{pi}} \rho_g v_{gx} (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} dS + \sum_i \int_{S_{pi}} (\boldsymbol{\sigma}_g \cdot \mathbf{n})_x dS. \end{aligned}$$

The left-hand side represents the rate of accumulation of momentum in the control volume. Terms on the right-hand side represent, in order, net momentum flux through the end plane,  $x$ -component of the forces on the end planes, the momentum flux into the control volume from the particle holes, and finally the sum of the  $x$ -direction forces acting at the particle holes. The viscous stress and momentum flux through the side walls of the control volume are assumed to be negligible. Thus the theory developed here is analogous to quasi one-dimensional gas dynamics. It is not inconsistent, however, to account for the effects of viscosity when considering the flow near the particle surface.

Proceeding as before, the area-averages are introduced. The right side is estimated by the mean-value theorem, and the equation is divided by  $\Delta x$  so that the limit  $\Delta x \rightarrow 0$  can be applied as a final step. There are two terms which occur because of momentum transfer between the particles and gas. The first is a drag from the integration of the forces acting at the gas-particle surface:

$$F_x \equiv \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \sum_i \int_{S_{pi}} (\boldsymbol{\sigma}_g \cdot \mathbf{n})_x dS. \quad (4.2)$$

The second interphase-momentum-transfer term accounts for momentum carried into the gas phase by mass vaporizing from the particles:

$$- \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \sum_i \int_{S_{pi}} \rho_g v_{gx} (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} dS. \quad (4.3)$$

The expression  $\rho_g (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n}$  is the mass flux of gas leaving a point on the particle surface, and  $v_{gx}$  is the  $x$ -component of the gas velocity (momentum per unit mass) at the particle surface. Since the viscous boundary condition requires that the tangential velocity components match across a phase discontinuity,  $v_{gx}$  is closely related to the velocity of the particles. If one assumes that the vaporization (or condensation) is uniform around the surface of the particle and that the particle surface is moving in uniform translation (no distortion of the particle), then the expression (4.3) is equal to

$$\psi \|v_{r,x}\|.$$

$\psi$  is the mass transfer term previously defined, and  $\|v_{r,x}\|$  is the area-averaged particle velocity.

The derivation of the momentum equation is continued by introducing the simplifications and definitions discussed above. In terms of area-averaged quantities, the momentum equation reads

$$\frac{\partial}{\partial t} (A_g \|\rho_g v_{gx}\|) = -\frac{\partial}{\partial x} (A_g \|\rho_g v_{gx}^2\| + A_g \|\sigma_{gxx}\|) + \psi \|v_{fx}\| - F_x. \quad (4.4)$$

The  $xx$ -component of the stress tensor would usually be replaced by the pressure. Equation (4.4) is next time-averaged and then (3.10) and (3.12) are used to invert the order of the time and area-averaging. Again it is assumed that fluctuation in the area  $A_g'$  and density  $\rho_g'$  are negligible compared to their respective time-averaged values, e.g.

$$\begin{aligned} \langle\langle A_g \|\rho_g v_{gx}^2\| \rangle\rangle &= \langle\langle A_g \rangle\rangle \langle\langle \|\rho_g v_{gx}^2\| \rangle\rangle \\ &= \langle\langle A_g \rangle\rangle \|\langle\rho_g v_{gx}^2\rangle\| \\ &= \langle\langle A_g \rangle\rangle \|\langle\rho_g\rangle \langle v_{gx}^2 \rangle + \langle\rho_g\rangle \langle v'_{gx} v'_{gx} \rangle\| \\ &= \langle\langle A_g \rangle\rangle \|\langle\rho_g\rangle\| \langle\langle v_{gx}^2 \rangle\rangle_{\rho} + \|\langle v'_{gx} v'_{gx} \rangle\|_{\rho}. \end{aligned}$$

The final momentum equation has dependent variables which are area-averages of (time-averaged local properties):

$$\begin{aligned} \frac{\partial}{\partial t} \langle\langle A_g \rangle\rangle \|\langle\rho_g\rangle\| \|\langle v_{gx} \rangle\|_{\rho} & \quad (4.5) \\ &= -\frac{\partial}{\partial x} \{ \langle\langle A_g \rangle\rangle \|\langle\rho_g\rangle\| \langle\langle v_{gx}^2 \rangle\rangle_{\rho} + \|\langle v'_{gx} v'_{gx} \rangle\|_{\rho} \} \\ &\quad -\frac{\partial}{\partial x} \{ \langle\langle A_g \rangle\rangle \|\langle p_g \rangle\| \} + \langle\langle \psi \rangle\rangle \|\langle v_{fx} \rangle\| - \langle\langle F_x \rangle\rangle. \end{aligned}$$

In this equation we have been able to display explicitly the area-average of the local Reynolds stress. The Reynolds stresses do not appear if the order of time and area-averaging are not interchanged. When the gas flow is turbulent, the Reynolds stress is, of course, very important, but even for laminar flow the inherently unsteady character of the flow means that this term is not zero. Later in the general discussion it will be shown that the Reynolds stresses can be neglected in dilute flows.

The velocity variable in the continuity equation was the area-average weighted by the density  $\|\langle v_{gx} \rangle\|_{\rho}$ . This variable also appears on the left side of the momentum equation above; however, on the right side, the velocity is squared before being averaged over the area  $\|\langle v_{gx}^2 \rangle\|_{\rho}$ . For simplicity one would like to relate these averages by

$$\|\langle v_{gx} \rangle\|_{\rho}^2 = \|\langle v_{gx}^2 \rangle\|_{\rho}. \quad (4.6)$$

This equation would be true if the velocity profile is relatively flat across the area chosen for averaging as in figure 7*b*. The parabolic shaped profile in figure 7*a* is an example where this would not be valid. In this case a factor might be inserted in (4.6) to account for the profile shape.

The particle density has been depicted as uniform in both figures 7*a* and 7*b*; this is necessary to satisfy the criteria for interchanging the averaging processes. Segré & Silberberg's (1962) experiments in Poiseuille flow show that the particles tend to migrate to a position 0.6 of the distance from the centre. Turbulent flow

studies by Soo (1965) found considerable changes in particle density across the tube. In these cases we cannot area-average across the entire tube but we could make a local area-average if the criteria (3.14) and (3.15) can be satisfied. In a way this is similar to the time-average where  $f$  is a function of  $t$  even though an average was taken from  $t$  to  $t + t_0$ . The local average depends upon the position in the cross-section about which  $A$  was taken, and an *a priori* prediction of the particle density or velocity profile would be a two-dimensional problem. This is outside the scope of the present discussion.

### 5. Flow properties from the particle-phase continuity and momentum equations

Equations for the particle phase may be obtained by two different approaches. The direct approach of considering the particle-phase control volume was taken by Panton & Oppenheim (1966) and is somewhat longer than the method used here. In this paper the equations will be derived for the overall gas-plus-particle mixture. Then equations for the particle-phase can be developed as in the preceding sections. This can be done since it was assumed that there is no mass, momentum, or energy associated with the interface between the gas and particle material.

The derivation of the continuity equation closely follows the development given in § 3. The cross-section of the control volume is  $A$  and integrals over this region are split into the sum of integrals over  $A_g$  and  $A_p$ . Applying the integral form of the continuity equation to the mixture control volume yields

$$\begin{aligned} \frac{d}{dt} \int_x^{x+\Delta x} \left( \int_{A_g} \rho_g dA + \int_{A_p} \rho_p dA \right) dx \\ = - \int_{A_g} \rho_g \mathbf{v}_g \cdot \mathbf{n} dA \Big|_x^{x+\Delta x} - \int_{A_p} \rho_p \mathbf{v}_p \cdot \mathbf{n} dA \Big|_x^{x+\Delta x}. \end{aligned}$$

The integral over  $A_p$  is actually the finite sum of integrals over the distinct particles. The summation sign will not be used but instead we will note that  $A_p$  is a disjoint region. Performing the same operations as in § 3 produces the continuity equation for the mixture:

$$\begin{aligned} \frac{\partial}{\partial t} (\langle A_g \rangle \|\langle \rho_g \rangle\| + \langle A_p \rangle \rho_p) \\ = - \frac{\partial}{\partial x} (\langle A_g \rangle \|\langle \rho_g \rangle\| \|\langle v_{gx} \rangle\| + \langle A_p \rangle \rho_p \|\langle v_{px} \rangle\|). \end{aligned} \quad (5.1)$$

In this equation the time and area-averages of properties of the particle material are defined analogously to those for the gas-phase. Also the density of the particle material has been assumed constant. Subtracting the gas-phase equation, (3.16), from (5.1) gives the final particle-phase continuity equation:

$$\frac{\partial}{\partial t} (\rho_p \langle A_p \rangle) = - \frac{\partial}{\partial x} (\rho_p \langle A_p \rangle \|\langle v_{px} \rangle\|) - \langle \psi \rangle. \quad (5.2)$$

The vapour added to the gas-phase from the particle holes appears in this equation as material leaving the particle phase. The velocity of the particle phase is

defined by forming local time averages of the velocity of the particle material at each point and then averaging over the cross-section.

The momentum equation will be considered next. The integral momentum equation (4.1), written for the mixture control volume, is

$$\begin{aligned} & \frac{d}{dt} \int_x^{x+\Delta x} \left( \int_{A_g} \rho_g v_{gx} dA + \int_{A_\ell} \rho_\ell v_{\ell x} dA \right) dx \\ &= - \int_{A_g} \rho_g v_{gx} (\mathbf{v}_g \cdot \mathbf{n}) dA \Big|_x^{x+\Delta x} + \int_{A_g} (\boldsymbol{\sigma}_g \cdot \mathbf{n})_x dA \Big|_x^{x+\Delta x} \\ & \quad - \int_{A_\ell} \rho_\ell v_{\ell x} (\mathbf{v}_\ell \cdot \mathbf{n}) dA \Big|_x^{x+\Delta x} + \int_{A_\ell} (\boldsymbol{\sigma}_\ell \cdot \mathbf{n})_x dA \Big|_x^{x+\Delta x}. \end{aligned} \quad (5.3)$$

In the last term  $(\boldsymbol{\sigma}_\ell \cdot \mathbf{n})_x$  is the  $x$ -component of the stress on the liquid within the particle at the plane  $x$  on  $x + \Delta x$ . It is reasonable to neglect the normal viscous stress and replace this by the pressure of the liquid. A typical particle cut by an end plane is shown in figure 6.

Another force, the surface tension, must also be associated with the last term although it does not appear explicitly. The surface tension is idealized to act at the interface of the gas and liquid in order to provide mechanical equilibrium of the curved surface. In reality it is a very thin region of high stress. Assume momentarily that the particle in figure 6 is stationary. Mechanical equilibrium requires that the integral of  $(\boldsymbol{\sigma}_\ell \cdot \mathbf{n})_x$  over the cross-section  $A_\ell$  plus the surface tension force be balanced by the external gas pressure times  $A_\ell$ . Hence the last term in (5.3) is not of the order of magnitude of  $A_\ell p_\ell$ , which might be quite high because of the high internal pressure of the droplet, but of order  $A_\ell p_g$ . After manipulations of (5.3), the last term will represent the pressure force for the particles.

The derivation of the momentum equation follows closely the path taken in §4. There are some differences in how the particle pressure is handled and therefore certain intermediate steps involving this term will be discussed. The area-averaging definition is introduced into (5.3), the equation divided by  $\Delta x$ , and  $\lim \Delta x \rightarrow 0$  is taken. After these operations, the particle pressure term is

$$\frac{\partial}{\partial x} (A_\ell \|p_\ell\|).$$

This is not just the average internal pressure but still requires that the surface tension be included.

Another question arises at this point. The term above is the gradient of the aggregate pressure (plus surface tension) within the particles. Now when a particle is accelerated there exists a pressure gradient within the particle in the direction of the acceleration. Does the term above then depend upon the state of acceleration of the particles? It is demonstrated in the appendix that the gradient within a single particle is not important and that the following holds:

$$\frac{\partial}{\partial x} (A_\ell \|p_\ell\|) = \frac{\partial}{\partial x} (A_\ell \|p_{ic}\|),$$

where  $p_{ic}$  is the average pressure across the cross-section (including surface tension) at the centre of the particle  $i$ . This equation says that instead of averaging the local pressure within a droplet one may substitute the average pressure of the central cross-section of the particle. As discussed before,  $p_{ic}$  is equal to the gas

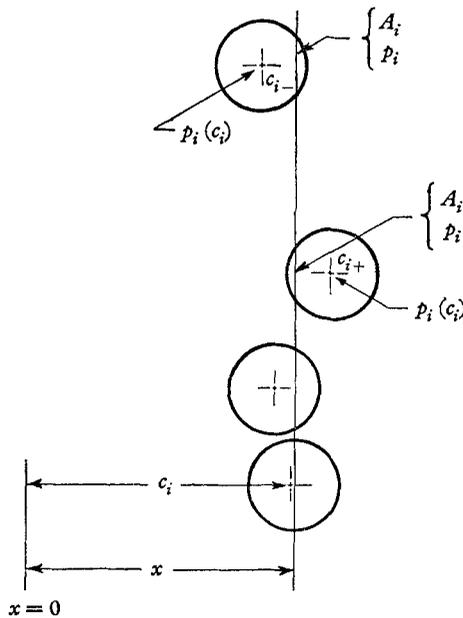
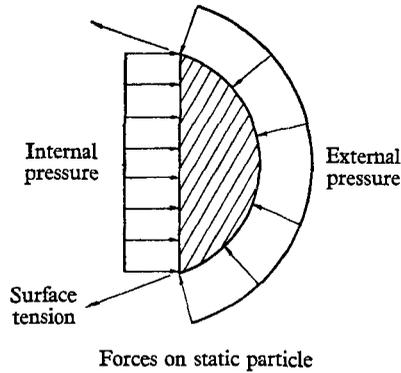


FIGURE 6. Particle-phase pressure.

pressure on the outside of the droplet when there is no motion. When there is motion, the average pressure around a particle, and thus  $p_{ic}$ , deviate slightly from the free-stream gas pressure. In any event, the particle pressure defined here is of the order of magnitude of the gas-phase pressure, its exact value depending upon the details of the local flow around each particle.

The final steps in deriving the momentum equation are to time-average the entire equation and then reverse the order of averaging. The result is

$$\begin{aligned} & \frac{\partial}{\partial t} (\langle A_g \rangle \| \langle \rho_g \rangle \| \| \langle v_{gx} \rangle \|_\rho + \langle A_e \rangle \rho_e \| \langle v_{ex} \rangle \|) \\ &= - \frac{\partial}{\partial x} \{ \langle A_g \rangle \| \langle \rho_g \rangle \| (\| \langle v_{gx} \rangle^2 \|_\rho + \| \langle v'_{gx} v'_{gx} \rangle \|) \} \\ & \quad - \frac{\partial}{\partial x} \{ \langle A_e \rangle \rho_e (\| \langle v_{ex} \rangle^2 \|_\rho + \| \langle v'_{ex} v'_{ex} \rangle \|) \} \\ & \quad - \frac{\partial}{\partial x} \{ \langle A_g \rangle \| \langle p_g \rangle \| + \langle A_e \rangle \| \langle p_{ic} \rangle \| \}. \end{aligned} \tag{5.4}$$

The particle-phase momentum equation is obtained by subtracting (4.5) from (5.4):

$$\begin{aligned} \frac{\partial}{\partial t} (\langle A_e \rangle \rho_e \| \langle v_{ex} \rangle \|) &= - \frac{\partial}{\partial x} \{ \langle A_e \rangle \rho_e (\| \langle v_{ex} \rangle^2 \|_\rho + \| \langle v'_{ex} v'_{ex} \rangle \|) \} \\ & \quad - \frac{\partial}{\partial x} \{ \langle A_e \rangle \| \langle p_{ic} \rangle \| \} + \langle F_x \rangle - \langle \psi \rangle \| \langle v_{ex} \rangle \|. \end{aligned} \tag{5.5}$$

In this equation the particles are viewed as a continuum which co-exists with the gas-phase. It includes two terms, the Reynolds stress and the particle pressure, which would not occur if one formulated the equation of motion for a ‘typical’ particle. In their paper Van Deemter & Van Der Laan (1961) proposed that some sort of particle-phase Reynolds stresses should be included for turbulent flow in analogy with single phase flow. Tien (1961) made the same argument for the turbulent transport of energy. In (5.5) the Reynolds stresses have been explicitly defined in terms of the local flow-field variables.

Two different area-averaged velocities appear in (5.5), as occurred in the gas-phase equation. In the formulation of a practical problem it is necessary to relate the different averages. The least complicated would be

$$\| \langle v_{ex} \rangle^2 \| = \| \langle v_{ex}^2 \rangle \|. \tag{5.6}$$

Figure 7 can be used to help obtain a physical understanding of when this is permissible. If there is a parabolic gas-phase velocity profile, one would expect a profile in the particle velocities also since the drag force strongly couples the two phases. In this case (5.6) would not be true without inserting a coefficient to account for the exact nature of the profile. The opposite situation is where a flat gas velocity profile would usually give a flat particle profile and (5.6) could be safely assumed. Turbulent gas flow would give rise to considerable fluctuation in the particle velocities and the product  $v'_{ex} v'_{ex}$  would be important. Even with laminar gas flow the particle velocities might fluctuate because slight variations in the particle sizes and shapes would cause different drag forces. This aspect needs careful consideration in a practical application since in many instances particles are nominally of a single size, but there is actually a large difference between the largest and smallest. It will be shown later that the Reynolds stresses may be dropped for dilute flows where the particles are in fact a uniform size.

Equation (5.5) can be simplified further assuming that (5.6) is valid. The left-hand side and the first term of the right are differentiated as products. Then the continuity equation is multiplied by  $\| \langle v_{\ell x} \rangle \|$  and subtracted. The result is

$$\begin{aligned} & \langle \langle A_{\ell} \rangle \rangle \rho_{\ell} \left\{ \frac{\partial}{\partial t} \| \langle v_{\ell x} \rangle \| + \| \langle v_{\ell x} \rangle \| \frac{\partial}{\partial x} \| \langle v_{\ell x} \rangle \| \right\} \\ & = - \frac{\partial}{\partial x} \{ \langle \langle A_{\ell x} \rangle \rangle \rho_{\ell} \| \langle v'_{\ell x} v'_{\ell x} \rangle \| + \langle \langle A_{\ell x} \rangle \rangle \| \langle p_{ic} \rangle \| \} + \langle \langle F_x \rangle \rangle. \end{aligned} \quad (5.7)$$

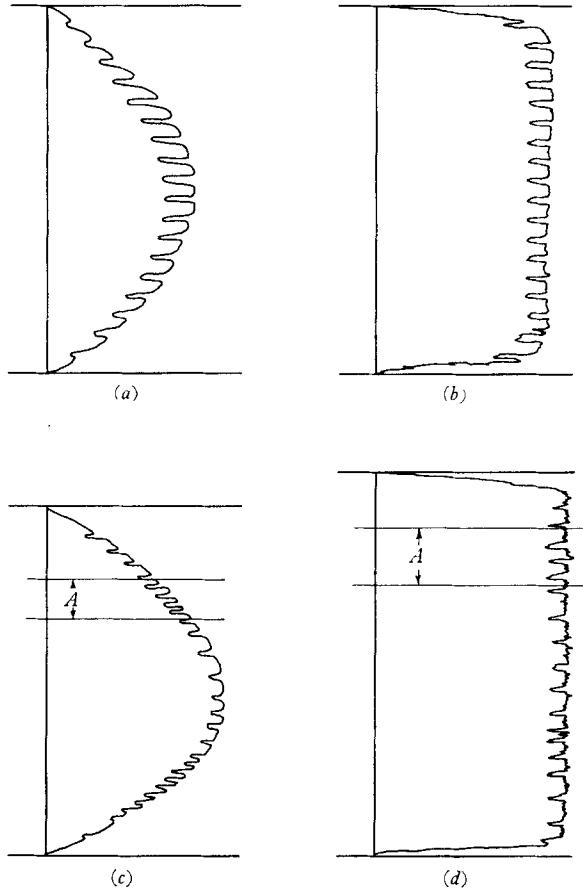


FIGURE 7. Velocity profiles.

Note that there is no mass-transfer effect in (5.7). From the point of view of a particle, the vaporizing material takes with it its momentum and does not affect the motion of the particle. If the vaporization had not been assumed uniform around the particle, there would be a net reaction force on the particle.

The particle and gas phase are considered as two interacting but co-existent continua in the continuity and momentum equations given above. Quasi one-dimensional flow was assumed; however, there was no restriction as to the volume occupied by the particle phase. The equations are valid for flows where the particles are dispersed. Criteria for 'dispersed flow' are that fluctuations in the

cross-section  $A'_g$  and particle cross-section  $A'_\ell$  are negligible compared to their respective time-averaged values. Another important assumption was that gas density fluctuations  $\rho'_g$  are small compared to the average value. The variables which describe the gas or particle phase were shown to be well-defined by time-averaging the local flow variable and then averaging across an area perpendicular to the flow.

## 6. Energy-equation properties for the particles and the mixture

The most useful form of an energy equation for the particles is the thermal-energy equation. In order to follow the procedure used in the preceding sections, the integral form of the thermal energy equation for a moving control volume is needed.

Leibnitz formula is applied to the internal energy per unit volume  $\rho e$ :

$$\frac{d}{dt} \int_{V(t)} \rho e dV = \int_{V(t)} \frac{\partial}{\partial t} (\rho e) dV + \int_{S(t)} \rho e \mathbf{v}_s \cdot \mathbf{n} dS. \quad (6.1)$$

Now the differential equation for the change in thermal energy is

$$\frac{\partial}{\partial t} (\rho e) = -\nabla \cdot (\rho e \mathbf{v}) - \nabla \cdot \mathbf{q} - p \nabla \cdot \mathbf{v} - \boldsymbol{\tau} : \nabla \mathbf{v}. \quad (6.2)$$

Equation (6.2) is substituted into (6.1), and the divergence theorem applied to yield the final result:

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho e dV = & - \int_{S(t)} \rho e (\mathbf{v} - \mathbf{v}_s) \cdot \mathbf{n} dS - \int_{S(t)} \mathbf{q} \cdot \mathbf{n} dS \\ & - \int_{V(t)} \{p(\nabla \cdot \mathbf{v}) + \boldsymbol{\tau} : \nabla \mathbf{v}\} dV. \end{aligned} \quad (6.3)$$

Thermal energy is not conserved. The last term shows two mechanisms to produce thermal energy. The first  $p(\nabla \cdot \mathbf{v})$  is that portion of compression work which increases the internal energy. Since the particles are incompressible,  $\nabla \cdot \mathbf{v} = 0$  and this term may be dropped. The second term  $\boldsymbol{\tau} : \nabla \mathbf{v}$  is the shear work which increases the internal energy. It is also permissible to neglect this term in low-speed incompressible flow.

The thermal-energy equation applied to the particle control volume is

$$\frac{d}{dt} \int_x^{x+\Delta x} \int_{A_\ell} \rho_\ell e_\ell dV = - \int_{S_{pt}} \{\rho_\ell e_\ell (\mathbf{v}_\ell - \mathbf{v}_s) + \mathbf{q}\} \cdot \mathbf{n}_\ell dS - \int_{A_\ell} \rho_\ell e_\ell \mathbf{v}_\ell \cdot \mathbf{n}_\ell dS \Big|_x^{x+\Delta x}. \quad (6.4)$$

Recall that  $S_{pt}$  stands for integration over all the particle surfaces. The outward normal from the particle is  $\mathbf{n}_\ell$  whereas the inward normal is  $\mathbf{n}$ .

The properties in the first term on the right-hand side of (6.4) are evaluated on the liquid side of the interface. It is more useful to have the gas-phase properties appear, since heat-transfer experiments would give the heat transfer from the gas to the particle  $q_g$ , not  $q_\ell$ . The conservation of total energy across a jump discontinuity such as the particle surface is

$$\begin{aligned} & \{\rho_\ell (\mathbf{v}_\ell - \mathbf{v}_s) (e_\ell + \frac{1}{2} v_\ell^2) + \mathbf{q}_\ell + p_\ell \mathbf{v}_\ell\} \cdot \mathbf{n}_\ell \\ & = - \{\rho_g (\mathbf{v}_g - \mathbf{v}_s) (e_g + \frac{1}{2} v_g^2) + \mathbf{q}_g + p_g \mathbf{v}_g\} \cdot \mathbf{n}. \end{aligned} \quad (6.5)$$

The change in kinetic energy of the vaporizing material is taken to be small when compared to the change in internal energy. Dropping these terms and rearranging so that the flow-work concept appears, one obtains

$$\begin{aligned} & \{\rho_\ell(\mathbf{v}_\ell - \mathbf{v}_s)(e_\ell + p_\ell/\rho_\ell) + \mathbf{q}_\ell\} \cdot \mathbf{n}_\ell \\ & = -\{\rho_g(\mathbf{v}_g - \mathbf{v}_s)h_g + q_g + (p_g - p_\ell)v_s\} \cdot \mathbf{n}, \end{aligned} \quad (6.6)$$

where  $h_g$  is the specific enthalpy of the gas phase. Because liquid densities are high, the liquid side flow work  $p_\ell/\rho_\ell$  can be dropped when compared to the internal energy. The extra term on the right is the work done to distort the surface. When (6.5) is substituted into (6.4), this term is

$$\int_{S_{pt}} (p_g - p_\ell) \mathbf{v}_s \cdot \mathbf{n} dS.$$

If the pressure change across the surface is uniform and the velocity constant, then this term is zero.

The heat flux vector in (6.5) consists of thermal conduction and transport by diffusion

$$\mathbf{q} = \mathbf{q}^{(c)} + \sum_k \rho_k(\mathbf{v}_k - \mathbf{v}_s)h_k,$$

where  $k$  indicates a particular species, and the summation is over all species present. Thus (6.5) now reads

$$[\rho_\ell(\mathbf{v}_\ell - \mathbf{v}_s)e_\ell + \mathbf{q}_\ell] \cdot \mathbf{n}_\ell = \{\rho_g(\mathbf{v}_g - \mathbf{v}_s)h_g + \sum_k \rho_k(\mathbf{v}_k - \mathbf{v}_g)h_k + \mathbf{q}_g^{(c)}\} \cdot \mathbf{n}.$$

This equation may be simplified further for the special case when only one species is vaporizing or condensing. Consider two species: an inert  $A$  and the vaporizing species  $B$ . The mass flow across the surface is entirely due to the mass flux of  $B$ :

$$\rho_g(\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} = \rho_B(\mathbf{v}_B - \mathbf{v}_s) \cdot \mathbf{n}. \quad (6.7)$$

Species  $A$  does not cross the surface. Actually  $A$  could stand for another mixture itself. Then

$$\rho_A(\mathbf{v}_A - \mathbf{v}_s) \cdot \mathbf{n} = 0. \quad (6.8)$$

The enthalpy of the gas mixture is given by

$$\rho_g h_g = \rho_A h_A + \rho_B h_B. \quad (6.9)$$

By using (6.7), (6.8) and (6.9), (6.6) will reduce to

$$\{\rho_\ell e_\ell(\mathbf{v}_\ell - \mathbf{v}_s) + \mathbf{q}_\ell\} \cdot \mathbf{n}_\ell = -\{\rho_g(\mathbf{v}_g - \mathbf{v}_s)h_{BS} + \mathbf{q}_g^{(c)}\} \cdot \mathbf{n}. \quad (6.10)$$

In this equation  $h_{BS}$  has replaced  $h_B$  in order to emphasize that it stands for the enthalpy of species  $B$  (the liquid) in the gaseous state at the droplet surface.

The particle thermal-energy equation (6.4) is modified by inserting (6.10):

$$\begin{aligned} \frac{d}{dt} \int_x^{x+\Delta x} \int_{A_\ell} \rho_\ell e_\ell dV & = \int_{S_{pt}} \rho_g(\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} h_{BS} dS + \int_{S_{pt}} \mathbf{q}_g^{(c)} \cdot \mathbf{n} dS \\ & \quad - \int_{A_\ell} \rho_\ell e_\ell \mathbf{v}_\ell \cdot \mathbf{n}_\ell dS \Big|_x^{x+\Delta x}. \end{aligned} \quad (6.11)$$

Manipulation of this equation follows the same path as above: the mean-value theorem used on the left, the area average defined, and the limit  $\Delta x \rightarrow 0$  performed after dividing by  $\Delta x$ . Heat transfer from the gas to the particles per unit flow length is called  $Q$ , where

$$Q = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_{S_{pi}} \mathbf{q}_g^{(e)} \cdot \mathbf{n} dS. \quad (6.12)$$

Energy transported with the mass flux is simplified by assuming the particle surface temperature is uniform; not only for each particle but among the particles themselves:

$$-\psi h_{BS} = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_{S_{pi}} \rho_g (\mathbf{v}_g - \mathbf{v}_s) \cdot \mathbf{n} h_{BS} dS. \quad (6.13)$$

With these operations the result is

$$\frac{\partial}{\partial t} (A_\ell \rho_\ell \|e_\ell\|) = -\frac{\partial}{\partial x} (A_\ell \rho_\ell \|v_{\ell x} e_\ell\|) + Q - \psi h_{BS}. \quad (6.14)$$

The final thermal-energy equation for the particles is found from (6.14) by time averaging and inverting the averaging order. The result is

$$\begin{aligned} \frac{\partial}{\partial t} (\langle A_\ell \rangle \rho_\ell \|\langle e_\ell \rangle\|) &= -\frac{\partial}{\partial x} (\langle A_\ell \rangle \rho_\ell \|\langle v_{\ell x} \rangle \langle e_\ell \rangle\|) \\ &\quad - \frac{\partial}{\partial x} (\langle A_\ell \rangle \rho_\ell \|\langle v'_{\ell x} v'_{\ell x} \rangle\|) + \langle Q \rangle - \langle \psi \rangle h_{BS}. \end{aligned} \quad (6.15)$$

In the first term on the right-hand side the convection of thermal energy appears  $\|\langle v_{\ell x} \rangle \langle e_\ell \rangle\|$ , whereas on the left  $\|\langle e_\ell \rangle\|$  occurs. Again for simplicity one would like to let

$$\|\langle v_{\ell x} \rangle \langle e_\ell \rangle\| = \|\langle v_{\ell x} \rangle\| \|\langle e_\ell \rangle\|. \quad (6.16)$$

This situation has been encountered previously in the momentum equations, and the arguments given for the validity of (5.6) can be adapted to this case showing that (6.16) is valid for flat velocity and temperature profile. The second feature of (6.15) is the turbulent transport of energy by fluctuations. This is the analogue of the Reynolds stresses in the momentum equation.

The substantial-derivative form of the thermal-energy equation is found by assuming (6.16), differentiating (6.15), and using the particle continuity equation:

$$\begin{aligned} \rho_\ell \langle A_\ell \rangle \left\{ \frac{\partial}{\partial t} \|\langle e_\ell \rangle\| + \|\langle v_{\ell x} \rangle\| \frac{\partial}{\partial x} \|\langle e_\ell \rangle\| \right\} &= -\frac{\partial}{\partial x} \{ \langle A_\ell \rangle \rho_\ell \|\langle v'_{\ell x} v'_{\ell x} \rangle\| \} \\ &\quad + Q - \psi \{ h_{BS} - \|\langle e_\ell \rangle\| \}. \end{aligned} \quad (6.17)$$

The last bracket is the heat of vaporization  $h_{fv}$  if two assumptions are made: the surface temperature is the same as the average particle temperature and the liquid internal energy is approximately equal to the enthalpy.

The second major topic of this section is the energy equation for the gas-particle mixture. The general equation for a control volume is

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho(e + \frac{1}{2}v^2) dV = & - \int_{S(t)} \rho(e + \frac{1}{2}v^2) (\mathbf{v} - \mathbf{v}_s) \cdot \mathbf{n} dS \\ & - \int_{S(t)} \mathbf{n} \cdot \{\mathbf{q} + p\mathbf{v} + \boldsymbol{\tau} \cdot \mathbf{v}\} dS. \end{aligned} \quad (6.18)$$

The mixture control volume is used in applying the equation. Integrals are split into portions referring to the liquid and portions referring to the gas. The established pattern is followed, utilizing the mean-value theorem, dividing by  $\Delta x$  and limiting  $\Delta x \rightarrow 0$ :

$$\begin{aligned} \frac{\partial}{\partial t} \{A_g(\|\rho_g e_g\| + \frac{1}{2}\|\rho_g v_g^2\|) + A_\ell \rho_\ell (\|e_\ell\| + \frac{1}{2}\|v_\ell^2\|)\} \\ = - \frac{\partial}{\partial x} \{A_g(\|\rho_g v_{gx} h_g\| + \frac{1}{2}\|\rho_g v_{gx} v_g^2\|) \\ + A_\ell \rho_\ell (\|v_{\ell x} h_\ell\| + \frac{1}{2}\|v_{\ell x} v_\ell^2\|) + \|q_{gx}\| + \mathbf{i} \cdot \|\boldsymbol{\tau} \cdot \mathbf{v}_g\|\}. \end{aligned}$$

The last two terms are the heat transported in the gas phase and the shear work; similar terms in the liquid have been dropped. The gas phase was taken as inviscid. Consistent assumptions of the same order are that no heat conduction or diffusion occurs so these gas-phase terms will be dropped also.

When the equation above is time-averaged and the order of averaging interchanged, a great many fluctuation terms will remain. Since the kinetic energy is frequently a small contribution, only fluctuations in thermal energy terms are retained. Then

$$\begin{aligned} \frac{\partial}{\partial t} \{ \langle A_g \rangle \langle \|e_g\| \rangle (\| \langle e_g \rangle \|_\rho + \frac{1}{2} \| \langle v_g \rangle^2 \|) + \langle A_\ell \rangle \rho_\ell (\| \langle e_\ell \rangle \| + \frac{1}{2} \| \langle v_\ell \rangle^2 \|) \} \\ = - \frac{\partial}{\partial x} \{ \langle A_g \rangle \langle \| \rho_g \rangle \| (\| \langle v_{gx} h_g \rangle \|_\rho + \| v'_{gx} h'_g \|) + \frac{1}{2} \| \langle v_{gx} \rangle \langle v_g \rangle^2 \| \} \\ + \langle A_\ell \rangle \rho_\ell (\| \langle v_{\ell x} \rangle \langle h_\ell \rangle \| + \| v'_{\ell x} h'_\ell \| + \frac{1}{2} \| \langle v_{\ell x} \rangle \langle v_\ell \rangle^2 \|) \}. \end{aligned} \quad (6.19)$$

Equation (6.19) is the total-energy equation for the entire mixture. All the different types of property definitions have been encountered and discussed previously. For steady flow the left-hand side vanishes and the right-hand side may be integrated to show that the bracketed term is constant. As a final remark, it is noted that a gas-phase energy equation could be derived by eliminating all the particle terms employing the particle-phase equations given previously.

## 7. Thermodynamic relations

The conservation equations must be supplemented by thermodynamic relations. The approach has been to define properties for the gas and particle phases as area-time averages of the local values. The result is that the particle and gas phases are co-existent but interacting continua. They interact when they are not in equilibrium.

Considering co-existent continua, the natural definition of density of the gas phase would be the mass of gas in a unit volume of space. This is given the symbol

$\delta_g$ . The relation between the 'continuum' gas density and the actual local gas density is

$$\delta_g A_x = \|\rho_g\| A_g.$$

The time-averaged form where fluctuations of  $\rho_g$  are negligible is

$$\langle \delta_g \rangle A_x = \|\langle \rho_g \rangle\| \langle A_g \rangle. \quad (7.1)$$

Another reason for the definition of  $\delta_g$  is to note that  $\|\langle \rho_g \rangle\|$  and  $\langle A_g \rangle$  always appear together in the conservation equations previously given. Since  $A_x$  is a constant, (7.1) will replace two variables by one. The same reasoning can be applied to the particle phase to define a continuum density  $\delta_p$ :

$$\left. \begin{aligned} \delta_p A_x &= \rho_e A_e, \\ \langle \delta_p \rangle A_x &= \rho_e \langle A_e \rangle. \end{aligned} \right\} \quad (7.2)$$

The following useful relation is valid when spherical particles with a number density  $n$  are used:

$$\delta_p = mn = \frac{4}{3}\pi r^3 \rho_e n = \rho_e A_e / A_x = (1 - A_g / A_x) \rho_e. \quad (7.3)$$

This equation will be used later in the general discussion. Although the particle material is incompressible, the particle 'continuum' is not since  $\delta_p$  changes with  $A_e$ .

The gas-phase pressure appears in the momentum equation (4.5). Like the density it is associated with the gas cross-section  $A_g$  since this is the area over which it acts. The 'continuum' gas-phase pressure is denoted by  $\pi_g$  and defined so that

$$\pi_g A_x = A_g \|p_g\|. \quad (7.4)$$

The particle-phase momentum has a similar structure; therefore the particle-phase pressure is defined as

$$\pi_p A_x = A_e \|p_{ic}\|. \quad (7.5)$$

The next question to resolve is how the new definitions are related in an equation of state. The gas phase is a perfect gas:

$$p_g = \rho_g R T_g.$$

This equation is time-averaged and area-averaged with the usual assumption that density fluctuations are small. Upon multiplying by  $\langle A_g \rangle / A_x$ , the final equation is

$$\langle \pi_g \rangle = \langle \delta_g \rangle R \|\langle T_g \rangle\|. \quad (7.6)$$

This is the same perfect-gas law in terms of the new continuum properties.

The caloric equation of state for a gas with constant specific heat is

$$h_g = c_p T_g + h_{REF}.$$

This equation may be averaged to yield

$$\|\langle h_g \rangle\|_\rho = c_p \|\langle T_g \rangle\|_\rho + h_{REF}. \quad (7.7)$$

Equation (7.7) completes the gas-phase thermodynamics.

The particle material is incompressible; therefore, there is no thermodynamic relation between the density and the pressure. The energy and the temperature are related with  $c$  denoting the specific heat:

$$e_{\ell} = cT_{\ell} + e_{REF}.$$

It is unnecessary to distinguish between the specific heat at constant pressure and at constant volume since they hardly differ. Averaging the equation above gives

$$\|\langle e_{\ell} \rangle\| = c \|\langle T_{\ell} \rangle\| + e_{REF}. \quad (7.8)$$

Thermodynamic equations for the mixture are not very useful; however, they will be discussed to demonstrate the consistency of the formalism. The pressure of an ideal mixture is the sum of the partial pressures:

$$\pi_m = \pi_p + \pi_g. \quad (7.9)$$

The density is the sum of the component densities:

$$\delta_m = \delta_p + \delta_g. \quad (7.10)$$

The enthalpy is given by  $\delta_m h_m = \delta_p h_{\ell} + \delta_g h_g$ . (7.11)

Equilibrium between the phases implies that the velocities and temperatures are the same and no mass transfer is occurring. When the droplets are stationary with respect to the gas, the average internal particle pressure is the same as the gas pressure as noted in §5. Thus

$$\pi_m = (A_{\ell}/A_x)p_g + (A_g/A_x)p_g = p_g.$$

The equilibrium density is given by (7.10), which does not change. In mixture thermodynamics the mass fractions are frequently introduced. They are simply the density ratios

$$\delta_p/\delta_m, \quad \delta_g/\delta_m.$$

Dividing (7.10) by  $\delta_m$  expresses the rule that the sum of the mass fractions is one.

The specific heats and temperatures may be introduced into equation (7.11). If the two phases are always in equilibrium, then an overall mixture specific heat may be defined. Since there is mass transfer between the phases as equilibrium is maintained, the mixture specific heat contains a term involving the heat of vaporization. For this reason, the definition is cumbersome and does not really simplify the work. If there is no mass transfer as for solid particles, then the usual formula for the specific heat of an ideal mixture results.

In most flow problems the fluid mechanics events are so fast that the phases are not in equilibrium. Analysis using equilibrium thermodynamics have generally disagreed with the experiments. For example, a controversy about the speed of sound in a mixture of steam and water droplets is discussed by Collingham & Firey (1963).

The particle phase pressure  $\pi_p$  and density  $\delta_p$  are both variables; however, there is no equation of state to relate them. The pressure  $\pi_p$  is indicative of the average pressure within the particles (including surface tension). This in turn depends upon the external pressure exerted on the droplet by the gas and also upon the details of the flow field around the particle. As previously noted in the absence of relative flow  $\pi_p$  is equal to  $\pi_g$ . It would seem reasonable to use this

assumption even for disequilibrium cases and ignore the flow field effect. This assumption takes the place of the missing equation of state.

The number density  $n$  and the mass of a particle  $m$  are not used explicitly in the conservation or thermodynamic equations. It is necessary to have a knowledge of the number density in order to formulate the interphase transport terms  $\psi$ ,  $F$  and  $Q$ . An equation for  $n$  and  $m$  can be obtained by substituting (7.1) and (7.3) into the continuity equation (5.2):

$$\langle m \rangle \left\{ \frac{\partial}{\partial t} \langle n \rangle + \frac{\partial}{\partial x} (\langle n \rangle \| \langle v_{fx} \rangle \|) \right\} + \langle n \rangle \left\{ \frac{\partial}{\partial t} \langle m \rangle + \| \langle v_{fx} \rangle \| \frac{\partial}{\partial x} \langle m \rangle \right\} = \psi / A_x.$$

Consider the special case where there is no mass transfer, then

$$\frac{\partial}{\partial x} \langle n \rangle + \frac{\partial}{\partial x} (\langle n \rangle \| \langle v_{fx} \rangle \|) = 0. \tag{7.12}$$

However, this equation does not depend upon  $m$ ; therefore, it must be true whether  $m$  is constant or not. If the flow is steady then (7.12) shows that the particle flux  $\langle n \rangle \| \langle v_{fx} \rangle \|$  is constant. Equation (7.12) expresses the conservation of the number of particles; if the particles break up or coalesce, then a source term would appear on the right side. Another equation is found by substituting (7.12) from the preceding equation:

$$\frac{\partial}{\partial t} \langle m \rangle + \| \langle v_{fx} \rangle \| \frac{\partial}{\partial x} \langle m \rangle = \frac{\psi}{\langle n \rangle A_x}.$$

Physically this is a statement that the mass of a particle changes because of the vaporization or condensation at the surface.

### 8. General discussion

Investigations of two-phase flow usually begin by assuming that properties such as the particle-phase velocity exist. A new approach was taken here, where the particles are viewed as macroscopic bodies, either solid or liquid, embedded in a gas. The conservation equations of continuum fluid mechanics are assumed to apply to the flow field locally, both within the particles and through the gas. Next, control volumes for each phase are defined and integral forms of the conservation equations applied. By inspecting the equations, the proper area-averaged properties may be defined so that they are meaningful terms in the physical conservation laws. Because the detailed flow is inherently unsteady, it was necessary to take the time average of the equations. The final step was to interchange the order of the time and area averages. This can only be done if certain criteria are satisfied. The final conservation equations in terms of the new properties contain several terms which have a different physical basis and interpretation than in previous work.

The assumption that the gas and liquid material are locally a continuum can be examined by considering the flow through a high expansion rocket nozzle. Starting at the throat the pressure is relatively high so that the mean free path in the gas is negligible compared to the particle diameter. The gas density drops as the flow expands and the mean free path assumes importance compared to a particle diameter. The conservation of material, momentum, and energy still

hold of course; however, the Navier–Stokes equations are questioned because temperature and velocity gradients compare with the mean free path. It turns out that Fourier’s heat conduction law and Newtonian viscosity can still be used in the conservation laws if only the boundary condition at the particle-gas interface are modified. This is the slip-flow region. As the gas continues to expand the free-molecule-flow region is encountered; the mean free path is large compared to the particle diameter. So far the major effect has been to change the nature of the gas-particle interactions. This is accounted for in the equations developed previously by changing the interphase-transport terms  $\psi$ ,  $F$  and  $Q$ . For instance, Crowe (1967) has recently discussed the drag coefficient of a particle in slip and free molecular flow. Another aspect of this example concerns the gas flow alone. In continuum flow the collisions between molecules continually redistributes the momentum. As the gas expands and the mean free path becomes large, collisions are less frequent and in the rarefied limit, thermodynamic equilibrium is not maintained. Hamel & Willis (1966) have studied theoretically the free expansion of a jet using kinetic theory and introducing the concept that the gas has two temperatures, parallel and perpendicular to the flow. When the situation occurs, the model of the gas phase previously given breaks down and the equations are invalid. Scott (1967) has reported experimental results which do not confirm the kinetic theory model, and this subject is a current topic in rarefied gas dynamics.

The properties of each phase were defined as the area-average of the time-averaged local property. Special definitions of the averages were needed since one phase does not occupy the entire volume. Consider the conceptual process of computing the particle velocity at a given location  $x$ . The plane at  $x$  intercepts the particles forming a great many circular regions. Within the circles the liquid has some instantaneous velocity. It also has a time-averaged velocity which was computed by noting the liquid velocity as several previous droplets passed this position. The particle-phase velocity is computed by making an area average over all the circles of the time-averaged liquid velocity. Other properties of both the gas and particles phases are defined in a similar way.

Conservation equations were developed where the variables were area-time averaged properties. During the manipulations it was necessary to reverse the order of averaging. The inversion could only be accomplished if certain criteria were met. In the gas-phase equations the criteria was that fluctuations in the gas cross-section were small; a similar criteria must be met by the particle fluctuations in the particle equations:

$$A'_p/\langle A_p \rangle \simeq 0, \quad A'_g/\langle A_g \rangle \simeq 0.$$

These criteria are not independent since  $A_p + A_g = A_x$ , a constant. It is illuminating to recast the criteria in terms of the number density. The instantaneous properties in (7.3) are replaced by their time average plus the fluctuations; this gives one equation. Next, (7.3) is time averaged and divided into the first equation to yield

$$\frac{\delta'_p}{\langle \delta_p \rangle} = \frac{n'}{\langle n \rangle} = \frac{A'_p}{\langle A_p \rangle} = \frac{A'_g/A_x}{1 - \langle A_g \rangle/A_x}. \quad (8.1)$$

This equation shows that the criteria  $A'_e/\langle A_e \rangle \simeq 0$  is directly equivalent to small fluctuations in the number density. Another statement would be that fluctuations in the 'continuum' particle density  $\delta_p$  are small. The equation above can be solved for the gas-phase criteria:

$$\frac{A'_g}{\langle A_g \rangle} = \frac{n' \langle A_e \rangle}{\langle n \rangle \langle A_g \rangle}. \tag{8.2}$$

This is the same relation to the number density modified by  $\langle A_e \rangle/\langle A_g \rangle$ . Even in non-dilute flows the portion of the cross-section occupied by particles is usually less than that occupied by the gas, so the gas fluctuation criteria is less severe than the particle criteria. In fact, in the so-called dilute flows  $\langle A_e \rangle/\langle A_g \rangle \rightarrow 0$  and the gas-phase criteria is satisfied.

The viewpoint that the particles and gas are co-existent continua has led previous workers to define the density of each phase based upon the entire mixture volume,  $\delta_p$  and  $\delta_g$  in the notation used here. A new aspect introduced here is the continuum pressures,  $\pi_p$  and  $\pi_g$  based on the total mixture cross-section. The gas-phase equation of state in three variables was shown to be the same form as the perfect gas law in the local variables. Furthermore, if one substitutes the relations for the  $\delta$  and  $\pi$  into the conservation equations of the preceding sections, then they formally appear to be two sets of conservation equations for single phases with source-like interaction terms  $\psi/A_x, F/A_x, Q/A_x$ . This substitution replaces the terms  $\rho_e, \rho_g, p_g, p_e, A_e$  and  $A_g$  with  $\pi_p, \pi_g, \delta_g$  and  $\delta_p$ . The variables are reduced by one and there is no longer any quantity to indicate whether the flow is dilute or not (i.e.  $\langle A_e \rangle/A_x$ ).

The particle-phase pressure  $\pi_p$  is interpreted physically as an average internal particle pressure. This is also a fresh insight. Previous workers have employed the interpretation that the particle pressure arises from random particle motion. This is inferred by analogy with the kinetic theory of a gas where random molecular momentum leads to the macroscopic pressure. The development of the particle momentum equation given here shows that  $\pi_p$  is the appropriate pressure and ascribes to the fluctuations their proper role as Reynolds stresses. If the kinetic-theory analogy were valid, it could logically be extended to the particle temperature since random molecular kinetic energy is associated with the macroscopic internal energy or temperature. It is clear that the analogy breaks down at this point since the internal energy of the particles is obviously dependent upon the internal particle temperature.

The area-averaged Reynolds stresses appeared in the momentum equations because the flow is inherently unsteady. Likewise terms accounting for the transport of thermal energy by fluctuations occur in the energy equations. These added unknowns are the price to be paid for dealing in averages instead of solving for the details of the flow. The proper assumptions about the fluctuation terms depend upon the specific problem being considered.

The manner in which Reynolds stresses come into the particle momentum equation is important. It gives a physical insight into the particle viscous effect in two- or three-dimensional flows where a particle velocity gradient exists. If a three-dimensional derivation were carried out, a Reynolds stress  $\|\langle v'_{ex} v'_{ey} \rangle\|$

coupling the fluctuations in two directions would occur. The fluctuation of the particle velocity in the  $y$ -direction transports  $x$ -direction momentum, the effective action of viscosity. Murray (1965) in his work on the motion of a gas bubble through a fluidized bed ('boiling') employed a particle-phase viscous effect. The formulation was attributed to unpublished work of Carrier and Cashwell. It was based on equating the shear stress with a rate of particle strain. Particle strain was analysed by analogy with solid mechanics using the configuration of the separated particle. The particle-phase viscosity derived on this basis is a property independent of the flow field. The physics of this model disagree with the concepts outlined above, i.e. that the particle viscous effect is a Reynolds stress phenomena, not truly viscous and dependent upon the nature of the particle flow fluctuations.

The co-existent gas and particle phases interact through the source terms  $\psi/A_x$ ,  $F/A_x$  and  $Q/A_x$ . For the case of dilute flows it seems reasonable to evaluate these terms using drag, heat and mass transfer coefficients for a single particle. However, shock tube experiments by Rudinger (1965) indicate a large discrepancy for the drag. These results are only tentative and may be caused by extraneous factors. In non-dilute flows the flow field of the particles influence each other and this must be accounted for in  $F$ . Particle-particle collisions are not accounted for in the theory presented here. They do not appear to be important even in non-dilute fluidized beds when one particle size is present. Marble (1964) has investigated theoretically collision effects in solid particles, and Crowe & Willoughby (1966) have recently formulated a problem where droplets of different sizes coalesce in a rocket nozzle.

Another comment on the source terms concerns the mass transfer. The momentum carried into the gas phase by vaporizing particle material is accounted for by  $\| \langle v_{\ell x} \rangle \| \langle \psi \rangle$ . If material were condensing from the gas phase, the sign of  $\psi$  would change. In either case the matter which is about to change phase is located at the surface of the particle and the velocity on either side of the surface is the same. This result disagrees with that of Lype (1965) and several other papers concerned with condensation on particles. For vaporization it is easy to see that the matter enters the gas phase with a velocity corresponding to the particle velocity. On the other hand, for condensation, the gas does not change into liquid while it moves at the gas-phase velocity. It is first slowed down as it passes through the boundary layer around the particle and because of the no-slip condition the gas is moving with the particle speed just prior to changing phase.

The remainder of the remarks will discuss some aspects of dilute flows. As noted previously the basic equations do not distinguish between dilute and non-dilute flows. The difference is that several terms drop out for dilute flows and a simpler problem results.

Dilute flow has come to mean a flow where the volume of the particles is small compared to the total volume  $\langle A_{\ell} \rangle / A_x \simeq 0$ . Recall that

$$\delta_p = (A_{\ell} / A_x) \rho_{\ell} = \frac{4}{3} \pi r^3 \rho_{\ell} n = (1 - A_g / A_x) \rho_{\ell}.$$

$\delta_p$  must be finite; otherwise the particle-phase problem vanishes. Since  $\langle A_{\ell} \rangle / A_x \rightarrow 0$ ,  $\rho_{\ell}$  must approach  $\infty$ . Looking at the next term, if  $\rho_{\ell} \rightarrow \infty$ , either

$r^3$  or  $n$  must go to 0. If  $r \rightarrow 0$  then drag, heat and mass transfer will vanish and there will be no interaction between the gas and particles. Thus the number density must vanish as  $\rho_p$  increases. The last terms show that  $A_g \rightarrow A_x$ .

The limiting forms of the continuum properties are readily found from the defining equations:

$$\begin{aligned} \delta_p &\rightarrow \delta_p, & \pi_g &\rightarrow 0, \\ \delta_g &\rightarrow \rho_g, & \pi_p &\rightarrow p_g. \end{aligned}$$

The major simplification is that the particle pressure drops out.

Next consider the Reynolds stresses. If the gas flow is turbulent, then the Reynolds stresses are important and must be dealt with. However, suppose that the flow is laminar and the fluctuations in the gas velocity occur because of the presence of the particles. Every time a particle passes the point under observation, a fluctuation in the gas velocity occurs. For the special case of the dilute limit, the number density approaches 0; therefore the gas-phase Reynolds stresses will vanish.

The particle-phase Reynolds stresses are also negligible for dilute flows but for slightly different reasons. Particle fluctuations are the result of fluctuations in the drag force. These variations would depend upon things like size and shape variations, local fluctuations in the gas flow, and other random phenomena. All of these phenomena are independent of the particle material density  $\rho_p$ ; therefore the magnitude of the fluctuating drag force is the same as the dilute limit  $\rho_p \rightarrow \infty$  is taken. Since the density of the particle increases, its response to the same fluctuating force will diminish. As  $\rho_p \rightarrow \infty$ , then the particle-phase Reynolds stresses will approach zero. These results together with those discussed above provide a considerable simplification of the equations for the special class termed dilute flows.

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## Appendix

It will be demonstrated that the particle-phase pressure force  $A_p \langle p_p \rangle$  may be computed by substituting the average pressure across the centre of the particle. We refer to figure 6 for an illustration of the nomenclature. By definition  $A_p \langle p_p \rangle$  is computed by choosing a plane at  $x$  and integrating the pressure of the particle material intercepted by this plane. Again we remark that the integral must include the surface tension force at the edge of the particle. Let the cross-section of the  $i$ th particle intercepted by a plane be  $A_i$  and the average pressure  $p_i$ . Thus

$$A_p \langle p_p \rangle \equiv \sum_i \int_{A_{pi}} p_p dA = \sum_i A_i p_i.$$

The  $x$ -co-ordinate of the centre of a particle will be called  $c_i$ . Expanding  $p_i$  in a Taylor's series about the centre of each particle yields

$$A_p \langle p_p \rangle = \sum_i A_i \left( p_i(c_i) + \left. \frac{dp_i}{dx} \right|_{c_i} (x - c_i) + \dots \right).$$

Now we make a distinction between particles on the right side of the plane +, and those on the left -. With this notation we have

$$A_{\ell}\langle p_{\ell}\rangle = \sum_{i+} A_{i+} \left( p_i(c_{i+}) + \left. \frac{dp_i}{dx} \right|_{c_{i+}} (x - c_{i+}) + \dots \right) \\ + \sum_{i-} A_{i-} \left( p_i(c_{i-}) + \left. \frac{dp_i}{dx} \right|_{c_{i-}} (x - c_{i-}) + \dots \right).$$

Numerous particles occupy the cross-section and it is assumed that one from each side can be paired together according to the distance from the particle centre to the plane. That is the particles are paired so that

$$(x - c_{i-}) = -(x - c_{i+}) = (x - c_i)_{\pm}.$$

The cross-sections of each pair are also the same

$$A_{i+} = A_{i-} = A_{i\pm}.$$

The pressure force is written as the summation over  $i \pm$  pairs

$$A_{\ell}\langle p_i \rangle = \sum_{i\pm} A_{i\pm} \left\{ p_i(c_{i+}) + p_i(c_{i-}) + (x - c_i)_{\pm} \left[ \left. \frac{dp_i}{dx} \right|_{c_{i+}} - \left. \frac{dp_i}{dx} \right|_{c_{i-}} + \dots \right] \right\}.$$

There is a possibility that one particle is left over; however, we accept this error. Two final assumptions about the flow will be made. First that the average centre pressure of the particles does not change appreciably over the distance of a particle diameter, i.e.

$$p_i(c_{i+}) \simeq p_i(c_{i-}) = p_{ic}(x).$$

This is essentially a requirement that the gas-phase pressure is approximately constant over the same distance. The second assumption is that the particle acceleration is approximately constant over the distance of a particle diameter. The pressure gradient within the particle characterizes this acceleration and so

$$\left. \frac{dp_i}{dx} \right|_{c_{i+}} = \left. \frac{dp_i}{dx} \right|_{c_{i-}}.$$

With these assumptions

$$A_{\ell}\langle P_{\ell} \rangle = \sum_{i\pm} 2A_{i\pm} p_{ic}(x) = A_{\ell}\langle p_{ic} \rangle.$$

Thus the particle-phase pressure is independent of the state of acceleration of the particles.

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